

The final is comprehensive (8-9 pages).  
There will be two pages on ch 9.

Ch. 9: For the final be able to

1. Solve separable diff. eq.
2. Use initial conditions & constants.
3. Set up and do ALL the applied problems from homework.

*Worried about applied problems?*

Pay attention today and Monday in lectures. Know the homework well. And go thru my review sheets and look at old finals.

## *Newton's Cooling Law Experiment*

Hot water is in the cup. We will try to predict the temp. at the end of class.

1<sup>st</sup> measurement:

Time = 10:32      Temp = 160°F

2<sup>nd</sup> measurement:

Time = 10:45      Temp = 143°F

*we'll use this later today*

## 9.4 Differential Equations Applications

### 1. Law of Natural Growth/Decay:

Assumption: "The rate of growth/decay is proportional to the function value."

$$\frac{dP}{dt} = kP \text{ with } P(0) = P_0$$

We solve this in general last time and got

$$P(t) = P_0 e^{kt}$$

Example:

A population has 500 bacteria at  $t=0$ .

After 3 hrs there are 8000 bacteria.

Assume the pop. grows at a rate proportional to its size.

Find  $B(t)$ .

$$\frac{dB}{dt} = k B$$

$$B(0) = 500$$

$$B(3) = 8000$$

$$\Rightarrow B(t) = B_0 e^{kt}$$

$$B(0) = 500 \Rightarrow B_0 e^0 = 500 \Rightarrow B_0 = 500$$

$$B(3) = 8000 \Rightarrow 500 e^{3k} = 8000$$

$$\Rightarrow e^{3k} = 16$$

$$\Rightarrow 3k = \ln(16)$$

$$\Rightarrow k = \frac{1}{3} \ln(16) \approx 0.9242$$

(RELATIVE GROWTH RATE = 92.42% per hr)

$$B(t) = 500 e^{\frac{1}{3} \ln(16) t}$$

Example:

The half-life of cesium-137 is 30 years. Suppose we start with a 100-mg sample. Find  $m(t)$ .

$$\frac{dm}{dt} = km$$

$$m(t) = m_0 e^{kt}$$

$$m(0) = 100 \Rightarrow m_0 = 100$$

$$m(30) = 50 \Rightarrow 100 e^{30k} = 50$$

$$\Rightarrow e^{30k} = \frac{1}{2}$$

$$\Rightarrow 30k = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow k = \frac{1}{30} \ln\left(\frac{1}{2}\right) \approx -0.0231$$

(relative growth rate = -2.31% per year)

$$m(t) = 100 e^{\frac{1}{30} \ln\left(\frac{1}{2}\right) t}$$

Example:

Bob deposits \$2000 into a savings account. The money grows at a rate proportional to its size (i.e. compound interest). The balance in 4 years is \$2100. Find the formula  $A(t)$  for the amount in his account in  $t$  years.

$$\frac{dA}{dt} = rA \Rightarrow A(t) = A_0 e^{rt}$$

$$A(0) = 2000 \Rightarrow A_0 = 2000$$

$$A(4) = 2100 \Rightarrow 2000 e^{4r} = 2100$$

$$\Rightarrow e^{4r} = \frac{2100}{2000} = 1.05$$

$$\Rightarrow 4r = \ln(1.05)$$

$$\Rightarrow r = \frac{1}{4} \ln(1.05)$$

$$\approx 0.0121975$$

(1.22% annual rate)

$$A(t) = 2000 e^{\frac{1}{4} \ln(1.05) t}$$

## 2. Newton's Law of Cooling:

Assumption: "The rate of temperature change is proportional to the difference between the temperature of the object and its surroundings."

$T(t)$  = temp of object

$T_s$  = temp of surroundings =  $70^\circ\text{F}$

Difference =  $T - 70$

$$\frac{dT}{dt} = k(T - 70)$$

$$\int \frac{1}{T-70} dT = \int k dt$$

$$\ln|T-70| = kt + C_1$$

$$|T-70| = e^{(kt+C_1)}$$

$$T-70 = \pm e^{C_1} e^{kt}$$

$$T(t) = 70 + C_2 e^{kt}$$

$$C_2 = \pm e^{C_1}$$

GENERAL SOLIDS

$$T(t) = 70 + C e^{kt}$$

10:32 :  $T(0) = 160$

$t=0$

$$\Rightarrow 70 + C e^0 = 160$$

$$\Rightarrow C = 90$$

10:45 :  $T(13) = 143$

$t=13$

$$\Rightarrow 70 + 90 e^{13k} = 143$$

$$\Rightarrow 90 e^{13k} = 73$$

$$\Rightarrow e^{13k} = \frac{73}{90}$$

$$\Rightarrow 13k = \ln\left(\frac{73}{90}\right)$$

$$\Rightarrow k = \frac{1}{13} \ln\left(\frac{73}{90}\right) \approx -0.016103$$

$$T(t) = 70 + 90 e^{-0.016t}$$

At end of class, 11:30 ( $t=48$ )

$$T(48) = 70 + 90 e^{-0.016(48)}$$

$$\approx 111.54696^\circ\text{F}$$

### 3. Mixing Problems:

Assume you have a vat of liquid that has a substance (a contaminant) entering at some rate and exiting at some rate, then

*"The rate of change of the contaminant is equal to the rate at which the contaminant is coming IN minus the rate at which it is going OUT."*

These problems typically look like:

$V$  = volume of the vat (liters)

$t$  = time (min)

$y(t)$  = amount in vat (kg)

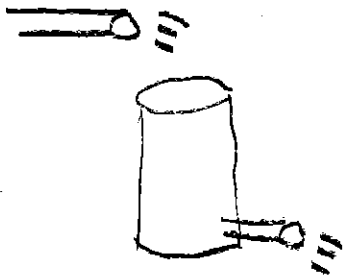
$\frac{dy}{dt}$  = rate (kg/min)

Thus,

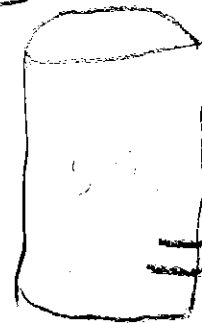
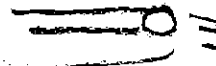
$$\frac{dy}{dt} = \text{Rate In} - \text{Rate out}$$

$$= \left( ? \frac{\text{kg}}{\text{L}} \right) \left( ? \frac{\text{L}}{\text{min}} \right) - \left( \frac{y}{V} \frac{\text{kg}}{\text{L}} \right) \left( ? \frac{\text{L}}{\text{min}} \right)$$

$$y(0) = ? \text{ kg}$$



IN



OUT

$y(t)$  = kg in here

$V$  = total volume

$$\frac{y(t)}{V}$$

= current concentration

Example:

Assume a 10 Liter vat contains 2kg of salt initially. A pipe pumps in salt water (brine) at 5 L/min with a concentration of 3 kg/L of salt.

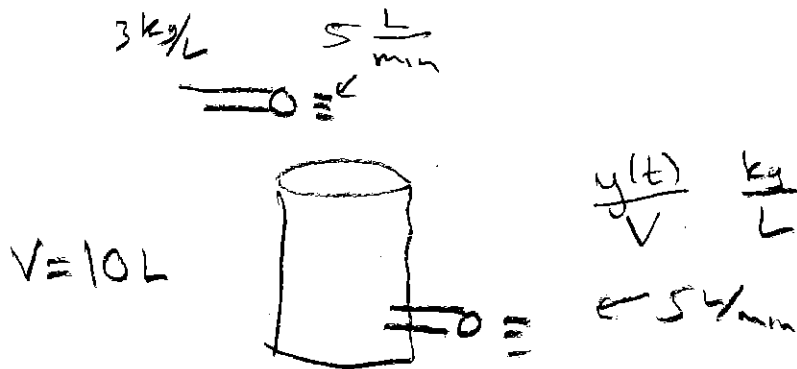
The vat is well mixed.

The mixture leaves the vat at 5L/min.

Let  $y(t)$  = the amount of salt in the vat at time  $t$ .

(a) Find  $y(t)$ .

(b) Find the limit of  $y(t)$  as  $n \rightarrow \infty$ .



$$y(t) = \text{kg of salt in vat}$$

$$\frac{dy}{dt} = \underbrace{15 \frac{\text{kg}}{\text{min}}}_{3 \cdot 5} - \frac{y}{10} \cdot 5 = \frac{\text{kg}}{\text{min}}$$

$$\frac{dy}{dt} = 15 - \frac{1}{2}y \quad y(0) = 2$$

$$\frac{1}{15 - \frac{1}{2}y} dy = 1 dt$$

$$\int \frac{2}{30 - y} dy = \int dt$$

$$-2 \ln |30 - y| = t + C_1$$

$$\ln |30 - y| = -\frac{t}{2} - \frac{t}{2} + C_1$$

$$|30 - y| = e^{(-\frac{t}{2} - \frac{t}{2} + C_1)}$$

$$30 - y = \pm e^{-\frac{t}{2}} \cdot e^{\frac{t}{2}}$$

$$C_2 = \pm e^{\frac{t}{2}}$$

$$y = 30 - C_2 e^{-\frac{t}{2}}$$

$$y(0) = 2 \Rightarrow 2 = 30 - C \Rightarrow C = 28$$

$$y(t) = 30 - 28e^{-\frac{t}{2}}$$

$$\lim_{t \rightarrow \infty} y(t) = 30 \text{ kg}$$

**Example:**

Assume a 100 Liter vat contains 5kg of salt initially. Two pipes (A & B) pump in salt water (brine).

Pipe A: Enters at 3L/min with a concentration of 4kg/L of salt.

Pipe B: Enters at 5L/min with a concentration of 2kg/L of salt.

The vat is well mixed.

The mixture leaves the vat at 8L/min.

Let  $y(t)$  = the amount of salt in the vat at time  $t$ .

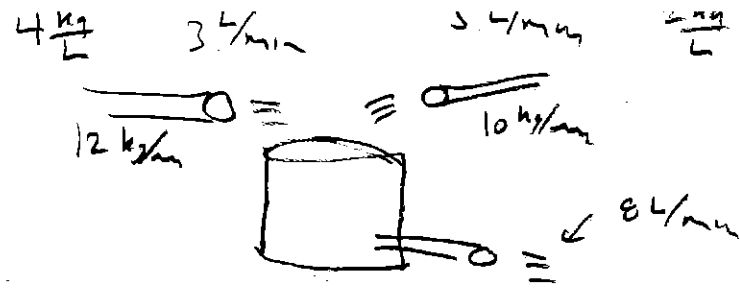
(c) Find  $y(t)$ .

(d) Find the limit of  $y(t)$  as  $t \rightarrow \infty$ .

$$y(t) = 275 + C e^{-\frac{2}{25}t}$$

$$y(0) = 5 \Rightarrow C = -270$$

$$\lim_{t \rightarrow \infty} y(t) = 275$$



$y(t)$  = kg of salt in vat

$$\frac{dy}{dt} = 22 - \frac{y}{100} \cdot 8$$

$$y(0) = 5$$

$$\frac{dy}{dt} = 22 - \frac{2}{25}y \quad y(0) = 5$$

$$\int \frac{1}{22 - \frac{2}{25}y} dy = \int 1 dt$$

$$-\frac{25}{2} \ln|22 - \frac{2}{25}y| = t + C_1$$

$$C_2 = -\frac{2}{25}C_1$$

$$\ln|22 - \frac{2}{25}y| = -\frac{2}{25}t + C_2$$

$$22 - \frac{2}{25}y = \pm e^{C_2 - \frac{2}{25}t} \quad C_3 = \pm e^{C_2}$$

$$\frac{2}{25}y = 22 - C_1 e^{-\frac{2}{25}t}$$

$$y = \frac{25}{2} \cdot 22 - \frac{25}{2} C_1 e^{-\frac{2}{25}t}$$